Properties of Real Numbers

For any real numbers ***a***, ***b***, ***c***:

## Communicative Properties:

**Addition**

**Multiplication**

## Associative Properties:

**Addition**

**Multiplication**

## Additive identity Property:

## Additive inverse Property:

## Multiplicative identity Property:

## Multiplicative inverse Property:

## Distributive Property:

Absolute Value

For any real number **a,**

When **a** is non-negative, the absolute value of **a** is **a**. When **a** is negative, the absolute value of **a** is the opposite, or additive inverse, of **a**.

is never negative; that is, for any real number **a**, **.**

Distance Between two Points on the Number line

For any real numbers **a** and **b**, the distance between **a** and **b** is:

or

Integers as Exponents

For any positive integer **n**

Where **a**, is the base and **n** is the exponent.

Zero and negative-integer exponents

For any non-zero real number **a** and any integer **m**,

and

For any non-zero numbers **a** and **b** and any integers **m** and **n**,

Properties of Exponents

For any real numbers **a** and **b** and any integers **m** and **n**, assuming **0** is not raised to a non-positive number:

## Product Rule:

## Quotient Rule:

## Power Rule:

## Raising a product to a power

## Raising a quotient to a power

Scientific notation

**Scientific notation** for a number is an expression of the type.

Where is in decimal notation, and **m** is an integer.

Rules For order of Operations

1. Do all calculations within grouping symbols before operations outside. When nested grouping symbols are present, work from the inside out.
2. Evacuate all exponential expressions.
3. Do all multiplication and divisions in order from left-to-right.
4. Do all additions and subtractions in order from left-to-right.

Polynomials In one variable

A **polynomial** **in one variable** is any expression of the type:

Where **n** is a non-negative integer ad are real numbers, called **coefficients**. The parts of a polynomial separated by plus signs are called **terms.** The **leading coefficient** is , and the **constant term** is . If , the **degree** of the polynomial is **n**. The polynomial is said to be written in **descending order**, because the exponents decrease from left-to-right.

Special Products of Binomials

## Square of a Sum

## Square of a Difference

## Product of a sum and a Difference

## Difference of Squares

## Sum of Cubes

## Difference of Cubes

Linear Equations

A **linear** **equation in one variable** is an equation that is equivalent to one of the form , where **a** and **b** are real numbers and **.**

Quadratic Equations

**A** quadratic equation **is an equation that is equivalent to one of the form where** a**,** b**, and** c **are real numbers and**

Equation solving Principles

For any real numbers **a**, **b**, **c**,

## The Addition Principle

Ifis true, then is true

## The Multiplication Principle

If is true, then is true.

## The Principle of Zero Products

If is true, then or , and if or , then .

## The Principle of Square Roots

If , then or

## Nth Root

A number **c** is said to be an **nth root** of **a** if **.**

Properties of Radicals

Let **a** and **b** be any real number or expressions for which the given roots exist. For any natural number **m** and **n** :

1. If is even,
2. If is odd,

The Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse”

Rational Exponents

For any real number **a** and any natural numbers **m** and **n**,, for which exists:

X-Intercept

An **x-intercept** is a point . To find **a**, let and solve for **x.**

Y-Intercept

A **y-intercept** is a point . To find **b**, let and solve for **y**.

Distance Formula

The **distance d** between any two point and is give by:

Midpoint Formula

If the endpoints of a segment are and **,** then the coordinates of the **midpoint** of the segment are

Equation of a Circle

The standard form of the equation of a circle with center and radius **r:**

Function

A **function** is a correspondence between a first set, called the **domain,** and a second set called the **range**, such that each member of the domain corresponds to exactly one member of the range.

Relation

A **relation** is a correspondence between a first set, called the **domain**, and a second set, called the **range**, such that each member of the domain corresponds to *at least one* member of the range.

Vertical Line Test

If it is possible for a vertical line to cross a graph more than once, then the graph is *not* the graph of a function.

Visualizing Domain and Range

**Domain** = the set of a function’s input, found on the horizontal **x-axis**.

**Range** = the set of a function’s output on the vertical **y-axis**.

Linear Function

A function is a **linear function** if it can be written as

Where **m** and **b** are constants.

If , the function is a **constant function** , if and , the function is the **identity function** .

Horizontal Lines and Vertical Lines

**Horizontal lines** are given by equations of the type or . (They are functions)

Vertical Lines

**Vertical lines** are given by equations of the type . (They are *not* functions)

Slope

The **slope m** of a line containing points and is give by

Horizontal Lines and Vertical Lines

If a line is horizontal, the change is **y** for any two points is **0** and the change in x is non-zero. Thus a horizontal line has slope 0. If a line is vertical, the change in x for any two points is **0**. Thus the slope is *not defined* because we cannot divide by 0.

Average Rate of Change

Slope can also be considered as an **average rate of change**. To find the average rate of change between any two data points on a graph, we determine the slope of the line that passes through the two points.

Slope-Intercept Equation

The linear function given by

is written in slope-intercept form. The graph of an equation in this form is a straight line parallel to . The constant **m** is called the slope, and the **y-intercept** is .

Point-Slope Equation

The **point-slope equation** of the line with slope **m** passing though is

Parallel Lines

Vertical lines are **parallel**. Non-vertical lines are parallel they have the same slope and different **y-intercepts**.

Perpendicular Lines

Two lines with slopes and are **perpendicular** the product of their slope is :

Lines are also **perpendicular** if one is vertical and the other is horizontal

Linear equation in one variable

A **linear equation in one variable** is in equation that can be expressed in the form , where **m** and **b** are real numbers and .

Equation-Solving Principles

For any real numbers **a**, **b**,and **c:**

## The Addition Principle

Ifis true, then is true

## The Multiplication Principle

If is true, then is true.

Simple interest rate

Zeros of Functions

An input **c** of a function is called a **zero** of the function if the output for the function is **0** when the input is **c**. That is **c**, is a zero of if .

Principles for solving Inequalities

For any real numbers **a**, **b**, **c**:

## Addition Principle for Inequalities

If is true, then is true.

## Multiplication Principle for Inequalities

1. If and are true, then is true.
2. If and are true, then is true.

When both sides of an inequality are multiplied by a negative number, the inequality sign must be reversed.

Similar statements hold for .

Increasing Functions

A function is said to be **increasing** on an *open interval*, if for all **a** and **b** in that interval, implies

Decreasing Functions

A function is said to be **decreasing** on an *open interval*, if for all **a** and **b** in that interval, implies

Constant Functions

A function is said to be **constant** on an *open interval*, if for all **a** and **b** in that interval,

Relative Maxima

Suppose that is a function for which exists for some in the domain of **.** Then

is a **relative maximum**  if there exists an *open interval* containing **c** such that , for all **x** where .

Relative Minima

Suppose that is a function for which exists for some in the domain of **.** Then

is a **relative minimum**  if there exists an *open interval* containing **c** such that , for all **x** where .

Properties of Functions

## Sum Property

## Difference Property

## Products Property

## Quotients Property

Composition of functions

The **composition function** , the **composition** of **f** and **g,** is defined as

Algebraic Test of Symmetry

**x-axis** : If replacing **y** with produces an equivalent equation, then the graph is **symmetric** with respect to the **x-axis.**

**y-axis** : If replacing **x** with produces an equivalent equation, then the graph is **symmetric** with respect to the **y-axis.**

**Origin** : If replacing **x** with and **y** with produces an equivalent equation, then the graph is **symmetric** with respect to the **origin**.

Even Functions   
If the graph of a function is symmetric with respect to the **y-axis**, we say that it is an **even function**. That is, for each **x** in the domain of

Odd Functions

If the graph of a function is symmetric with respect to the **origin**, we say that it is an **odd function**. That is, for each **x** in the domain of

Vertical Translation

For

the graph of is the graph of shifted up **b** units.

the graph of is the graph of shifted down **b** units.

Horizontal Translation

For

the graph of is the graph of shifted right **d** units.

the graph of is the graph of shifted left **d** units.

Reflections

The graph of is the **reflection** of the graph across the **x-axis**

The graph of is the **reflection** of the graph across the **y-axis**

If a point is on the graph **,** then is on the graph of **,** and is on the graph

Vertical Stretching and Shrinking

The graph of can be obtained from the graph by

Stretching vertically for , or

Shrinking vertically for .

For , the graph is also reflected across the **x-axis**.

(The **y-coordinates** of the graph can be obtained by multiplying the **y-coordinates** of by .

Horizontal Stretching and Shrinking

The graph of can be obtained from the graph of by

Shrinking horizontally for , or

Stretching horizontally for .

For , the graph is also reflected across the **y-axis**.

(The **x-coordinates** of the graph can be obtained by dividing the **x-coordinates** of the graph of by .

Direct Variation

If a situation gives rise to a linear function , or , where **k** is a *positive* constant, we say that we have **direct variation**, or that **y varies directly as x**, or that **y is directly proportional to x**. The number **k** is called the **variation constant**, or the **constant of proportionality**.

Inverse Variation

If a situation gives rise to a function , or , where **k** is a *positive* constant, we sat that we have **inverse variation**, or that **y varies inversely as x**, or that **y is inversely proportional to x**. The number **k** is called the **variation constant**, or the **constant of proportionally**.

Combined Variation

y varies **directly as the nth power of x** if there is some positive constant **k** such that

y varies **inversely as the nth power of x** if there is some positive constant **k** such that

y varies **jointly as x and z** if there is some positive constant **k** such that

The Number i

The number *i* is defined such that

and

Complex Numbers

A **complex number** is a number of the form , where **a** and **b** are real numbers. The number **a** is said to be the **real part** of , and the number **b** is said to be the **imaginary part** of **.**

Conjugate of a Complex Number

The **conjugate** of a complex number is . The number and are **complex conjugates**.

Quadratic Equations

A **quadratic equation** is an equation that can be written in the form

Where **a**, **b**, and **c** are real numbers.

Quadratic Functions

A **quadratic function**  is a function that can be written in the form

Where **a**, **b**, and **c** are real numbers.

Equation-Solving Principles

**The Principles of Zero Products:** If is true, then or , then .

**The Principle of Square Roots:** If , then or .

The Quadratic Formula

The solution of , are given by

Discriminant

For , where **a**, **b**, and **c** are real numbers:

Graphing Quadratic Functions

The graph of the function is a parabola that

* Opens up if and down if
* Has as the vertex
* Has as the axis of symmetry
* Has as a minimum value (output) if
* Has as a maximum value if

Vertex of a Parabola

The **vertex** of a graph of is

The Principle of Powers

For any positive integer **n:**

If is true, then is true

Equations with Absolute Value

For and an algebraic expression **X**:

Is equivalent to

or

Polynomial Function

A **polynomial function** P is given by

Where the coefficients are real numbers and the exponents are whole numbers.

Domain of a Polynomial Function

The *domain* of a polynomial functions is the set of all real numbers,

Even and Odd Multiplicity

If , is a factor of a polynomial function and is not a factor and:

* is odd, then the graph crosses the **x-axis** at
* is even, then the graph is tangent to the **x-axis** at

Graphing Polynomial Functions

If is a polynomial function of degree , then the graph of the function has:

* At most real zeros, and thus at most  **x-intercepts**
* At most turning points

(Turning points on a graph, also called **relative maxima** and **minima**, occur when the function changes from decreasing to increasing or from increasing to decreasing.)

To Graph a Polynomial Function

1. Use the leading-term test to determine the end behavior.
2. Find the zeros of the function by solving . Any real zeros are the first coordinates of the **x-intercepts**.
3. Use the **x-intercepts** (zeros) to divide the **x-axis** into intervals, and choose a test point in each interval to determine the sign of all function values in that interval.
4. Find . This gives the **y-intercept** of the function.
5. If necessary, find additional function values o determine the general shape of the graph and then draw the graph.
6. As a partial check, use the facts that the graph has at most  **x-intercepts** and at most turning points. Multiplicity of zeros can also be considered in order to check where the graph crosses or is tangent to the **x-axis.** We can also check the graph with a graphing calculator.

The Intermediate Value Theorem

For any polynomial function with real coefficients, suppose that for , and are opposite signs. Then the function has a real zero between and .

The intermediate value theorem *cannot* be user to determine whether there is a real zero between and when and have the *same* sign.

The Remainder Theorem

If a number is substituted for in the polynomial **,** then the result is the remainder that would be obtained by dividing by . That is, if , then

The Factor Theorem

For a polynomial , if , then is a factor of .

The Fundamental Theorem of Algebra

Every polynomial function of degree , with , has at least one *zero* in the set of complex numbers.

Polynomial function of degree n

Every polynomial function of degree , with , can be factored into linear factors (not necessarily unique); that is

Non-real Zeros:

If a complex number , is a zero of a polynomial function with *real coefficients*, then its conjugate, , is also a zero. For example, if is a zero of a polynomial function , with *real coefficients*, then its conjugate , is also a zero. (Non-real zeros occur in conjugate pairs.)

Irrational Zeros: , is not a Perfect Square

If , where ,, and are rational and is not a perfect square, is a zero if a polynomial function with *rational coefficients*, then its conjugate, is also a zero. For example, if is a zero of a polynomial function with rational coefficients, then its conjugate, , is also a zero. (Irrational zeros occur in conjugate pairs.)

The Rational zero Theorem

Let

Where all the coefficients are integers. Consider a rational number denoted by , where and are relatively prime (having no common factor besides and ). If is zero, then is a factor of and is a factor of

Descartes’ Rule of Signs

Let, written in descending order or ascending order, be a polynomial function with real coefficients and a non-zero constant term. The number of positive real zeros of is either:

1. The same as the number of variations of sign in , or
2. Less than he number of variations of sign in by a positive even integer.

The number of negative real zeros of is either:

1. The same as the number of variations of sign in , or
2. Less than the number of variations of sign in by a positive even integer.

A zero of multiplicity must be counted times.

Rational Function

A **rational function** is a function that is quotient of two polynomials. That is,

Where and are polynomials and where is not the zero polynomial. The domain of consists of all inputs for which

Determining Vertical Asymptotes

For a rational function **,** where and are polynomials with no common factors other than constants, if is a zero of the denominator, then the line is a vertical asymptote for the graph of the function.

Determining a Horizontal Asymptote

* When the numerator and the denominator of a rational function have the same degree, the line is the horizontal asymptote, where and are the leading coefficients of the numerator and the denominator, respectively.
* When the degree of the numerator of a rational function id less than the degree of the denominator, the **x-axis**, or , is the horizontal asymptote.
* When the degree of the numerator of a rational function is greater than the degree of the denominator, there is no horizontal asymptote.

Occurrence of Line as Asymptotes of Rational Functions

For a rational function , where and have no common factors other than constants

**Vertical asymptotes** occur at any **x-values** that make the denominator 0.

**The x-axis is the horizontal asymptote** occurs when the numerator and the denominator have the same degree.

**An oblique asymptote** occurs when the degree of the numerator is 1 greater than the degree of the denominator.

There can be only one horizontal asymptote or one oblique asymptote and never both.

An asymptote is *not* part of the graph of the function.

Crossing an Asymptote

* The graph of a rational function *never crosses* a vertical asymptote.
* The graph of a rational function *might* *cross* a horizontal asymptote but does not necessarily do so.

Graph a rational Function

To graph a rational function , where and have no common factor other than constants:

1. Find any real zeros of the denominator. Determine the domain of the function and sketch any vertical asymptotes.
2. Find the horizontal asymptote or the oblique asymptote, if there is one, and sketch it.
3. Find any zeros of the function. The zeros re found by determining the zeros of the numerator. These are the first coordinates of the **x-intercepts** of the graph.
4. Find . This give the **y-intercept**, , of the function.
5. Find other function values to determine the general shape. Then draw the graph.

Solve Polynomial inequality

To solve a polynomial inequality

1. Find an equality withon one side and on the other.
2. Change the inequality symbol to an equals sign and solve the related equation; that is solve .
3. Use the solutions to divide the **x-axis** into intervals. Then select a test value from each interval and determine the sign of the polynomial on the interval.
4. Determine the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. Include the endpoints of the intervals in the solution set if the inequality symbol is or .

Solve Rational Inequality

To solve a rational inequality:

1. Find an equivalent with on one side.
2. Change the inequality symbol to an equals sign and solve the related equation
3. Find the values of the variable for which the related rational function is not defined.
4. The numbers found in steps (2) and (3) are called **critical values**. Use the critical values to divide the **x-axis** into intervals using an **x-value** from the interval or the graph of the equation.
5. Select the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. If the inequality symbol is or , then the solutions to step (2) should be included in the solution set The **x-values** found in step (3) are never included in the solution set.

Inverse Relation

Interchanging the first and second coordinates of each ordered pair in a relation produces the **inverse relation**.

Inverse Relation

If a relation is defined by an equation, interchanging the variables produces an equation of the **inverse relation**.

One-to-One Functions

A function is **one-to-one** if different inputs have different outputs—that is ,

If then

Or, a function is **one-to-one** if when the outputs are the same, the inputs are the same—that is,

If then

One-to-One Functions AND Inverses

* If a function is one-to-one, then its inverse is a function.
* The domain of a one-to-one function is the range of the inverse .
* The range of a one-to-one function is the domain of the inverse function .
* A function that is increasing over its entire domain or is decreasing over its entire domain is a one-to-one function.

Horizontal Line Test

It is possible for a horizontal line to intersect the graph of a function more than once, then the function is *not* one-to-one and its inverse is *not* a function.

Obtaining a Formula for an Inverse

If a function is one-to-one, a formula for its inverse can generally be found as follows:

1. Replace with .
2. Interchange and **.**
3. Solve for .
4. Replace with .

Graph of

The graph of is a reflection of the graph of across the line .

Inverse Functions and Composition

If a function is one-to-one, then is the unique function such that each of the following holds:

For each in the domain of

For each in the domain of

Exponential Function

The function , where is a real number, and , is called the **exponential function, base a**

Logarithmic Function, Base 2

“,” read “the logarithm, base 2, of ,” means “the power to which we raise 2 to get .”

Logarithmic Function, Base a

We define as that number such that , where and is a *positive* constant other than .

Common Logs

and

For any logarithmic base **.**

Converting Between Exponential Equations and Logarithmic Equations

Natural Logarithms

and

The change-of-Base Formula

For any logarithmic bases and , and any positive number ,

Logarithmic Product Rule

For any positive numbers and and any logarithmic base ,

(The logarithm of a product is the sum of the logarithms of the factors.)

Logarithmic Power Rule

For any positive number , any logarithmic base , and any real number .

(The logarithm of a power ofis the exponent times the logarithm of .)

Logarithmic Quotient Rule

For any positive numbers and and any logarithmic base ,

(The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.)

The logarithm of a Base to a Power

For any base and any real number ,

(The logarithm, base, of to a power is the power.)

A vase to a Logarithmic Power

For any base and any real number ,

Base-Exponent Property

For any ,

Property of Logarithmic Equality

For any **,** ,and ,

Growth Rate and Doubling Time

The **growth rate**  and the **doubling time**  are related by

or

or

Decay Rate and Half-Life

The decay **rate**  and the half-life are related by

or

or

Converting From Base b to Base e

Trigonometric Function Values of n Acute Angle

Let be an acute angle of a right triangle. Then the six trigonometric functions of are as follows

Reciprocal Functions

Trigonometric Function Value

The trigonometric function values of depend only on the measure of the angle, *not* the size of the triangle.

Co-function Identities

Trigonometric Functions of Any Angle

Suppose that is any point other than the vertex on the terminal side of any angle in standard position, and is the radius, or distance from the origin to . Then the trigonometric functions are defined as follows:

Reference Angle

The **reference angle** for an angle is the acute angle formed by the terminal side of the angle and the **x-axis**.

Converting Between Degree Measure and Radian Measure

To convert from degree to radian measure, multiply by

To convert from radian to degree measure, multiply by

Radian Measure

The **radian measure**  of a rotation is the ratio of the distance traveled by a point at a radius from the center of rotation, to the length of the radius .

When we are using the formula  must be in radians and and must be expressed in the same unit.

Linear Speed in Terms of Angular Speed

The **linear speed** of a point a distance from the center of rotation is given by

Where is the **angular speed** in radians per unit of time.

Linear Speed in Terms of Angular Speed

For the formula , the units of distance for and must be the same, must be in radians per unit of time, and the units of time for and must be the same.

Basic Circular Functions

For a real number that determines a point on the unit circle:

Domain and Range of Sine and Cosine Functions

The *domain* of the sine function and the cosine function is

The *range* of the sine function and the cosine function is

Periodic Function

A function is said to be **periodic** if there exists a positive constant such that

For all n the domain of . The smallest such positive numberis called the period of the function.

Sine Function

1. Continuous
2. Period:
3. Domain:
4. Range:
5. Amplitude:
6. Odd:

Cosine Function

1. Continuous
2. Period:
3. Domain:
4. Range:
5. Amplitude:
6. Odd:

Tangent Function

1. Period:
2. Domain: except , where is an integer
3. Range:

Cotangent Function

1. Period:
2. Domain: except , where is an integer
3. Range:

Cosecant Function

1. Period:
2. Domain: except , where is an integer
3. Range:

Secant Function

1. Period:
2. Domain: except , where is an integer
3. Range:

Amplitude

The **amplitude** of the graphs of

and

is

Period

The **period** of the graphs of and is

Phase Shift

The **phase shift** of the graphs

and

Is

Transformation of Sine Functions and Cosine Functions

To graph

and

Follow these steps in the order in which they are listed

1. Stretch or shrink the graph horizontally according to
   1. , stretch horizontally
   2. shrink horizontally
   3. reflect across **y-axis**
2. Stretch or shrink the graph vertically according to .
   1. , stretch vertically
   2. shrink vertically
   3. reflect across **x-axis**
3. Translate the graph horizontally according to
   1. , units to the left
   2. , units to the right

The *phase shift* is

1. Translate the graph vertically according to
   1. , units down
   2. , units up

Basic Pythagorean Identities

Pythagorean Identities

Pythagorean identities Equivalent Forms

Sum and Difference Identities

There are six identities here

Co-function Identities

Co-function Identities for Sine and Cosine

Double-Angle Identities

Half-Angle Identities

Product-To-Sum Identities

Sum-To-Product Identities

Inverse Trigonometric Functions

, where

Domain:

Range:

, where

Domain:

Range:

, where

Domain:

Range:

Composition of Trigonometric Functions

, for all in the domain of

for all in the domain of

for all in the domain of

Special Cases

for all in the range of

for all in the range of

for all in the domain of

Sine Function

1. Domain:
2. Range:

Cosine Function

1. Domain:
2. Range:

Tangent Function

1. Domain: except odd
2. Range:

Cosecant Function

1. Domain: except
2. Range:

Secant Function

1. Domain: except odd
2. Range:

Cotangent Function

1. Domain: except
2. Range:

Arcsine Function

1. Domain:
2. Range:

Arccosine Function

1. Domain:
2. Range:

Arctangent Function

1. Domain:
2. Range:

Arccosecant Function

1. Domain:
2. Range:

Arctangent Function

1. Domain:
2. Range:

Angle-Angle-Side (AAS)

Two angles of a triangle and a side opposite one of them are known.

Angle-Side-Angle (ASA)

Two angles of a triangle and the included sides are known.

Side-Side-Angle (SSA)

Two sides of a triangle and an angle opposite one of them are known. (In this case, there may be no solution, one solution, or two solutions. The latter is known is the ambiguous case.)

Side-Angle-Side (SAS)

Two sides of a triangle and the included angle are known.

Side-Side-Side (SSS)

All three sides of the triangle are known.

The Law of Sines

In any triangle

The Area of a Triangle

The area of any is one-half of the product of the lengths of two sides and the sine of the included angle:

The Law of Cosines

In any triangle

Thus, in any triangle, the square of a side is the sum of the squares of the other two sides, minus twice the product of those sides and the cosine of the included angle. When the included angle is , the law of cosines reduces the Pythagorean theorem.

Use The Law of Sines

Use the law of sines for:

AAS

ASA

SSA

Use The Law Of Cosines

SAS

SSS

The law of cosines can also be used for the SSA situation, but since the process involves solving a quadratic equation, it is not included.

Absolute Value of a Complex Number

The **absolute value of a complex number**

Trigonometric Notation For Complex Numbers

Complex Numbers: Multiplication

For any complex numbers and

Complex Numbers: Division

For any complex numbers and

Demoivre’s Theorem

For any complex number and any natural number **,**

Roots Of Complex Numbers

The roots of a complex number ,, are given by

Where

Plot points on a polar graph:

To plot points on a polar graph

1. Locate the directed angle .
2. Move a directed distance from the pole. If , move along ray . If , move in the opposite direction of ray.

Vectors

A **vector** in the plane is a directed line segment. Two vectors are **equivalent** if they have the same *magnitude* and the same *direction*.

Component Form of a Vector

The **component form** of with and is

Length Of a Vector

The **length**, o **magnitude**, of a vector is given by

Equivalent Vectors

Let and . Then

If and only if and

Scalar Multiplication

For any real number and a vector , the **scalar product** of and is

The vector is a **scalar multiple** of the vector

Vector Addition

If and . Then

Vector Subtraction

If and . Then

Properties of Vector Addition and Scalar Multiplication

For all vectors and and for all scalars and :

Dot Product

The **Dot product** of two vectors and is

(Note that is a *scalar*, not a vector.)

Angle Between Two Vectors

If is the angle between two *non-zero* vectors and , then

Solving Systems of Equations in Three Variables

1. Interchange any two equations
2. Multiply on both sides of the equations by a non-zero constant
3. Add a non-zero multiple of one equation to another equation.

Row-Equivalent Operations

1. Interchange any two rows.
2. Multiply each entry in a row by the same non-zero constant
3. Add a non-zero multiple of one to another row.

Row-Echelon Form

1. If a row does not consist entirely of , then the first non-zero element in the row is a (called a **leading** ).
2. For any two successive non-zero rows, the leading in the lower row is farther to the right than the leading in the higher row.
3. All the rows consisting entirely of are at the bottom of the matrix.

If a fourth property is also satisfied, a matrix is said to be in **reduced row-echelon form**:

1. Each column that contains a leading has everywhere else.

Addition and Subtraction of Matrices

Given two matrices and , their sum is

and their difference is

Scalar Product

The **scalar product** of a number and a matric is the matrix denoted , obtained by multiplying each entry of by the number . The number is called a **scalar**.

Properties of Matrix Addition and Scalar Multiplication

For any matrices and and any scalars and :

## Communicative Property of Addition

## Associative Property of Addition

## Associative Property Scalar Multiplication

## Distributive Property

There exists a unique matrix such that:

## Additive Identity Property

There exists a unique matrix such that:

## Additive Inverse Property

Matrix Multiplication

For any matrix and an matrix the **product**  is an matrix where

Matrix Multiplication

Note that you can multiply two matrices only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

Properties of Matrix Multiplication

For matrices and assuming that the indicated operations are possible:

## Associative Property of Multiplication

## Distributive Property

## Distributive Property

Identity Matrix

For any positive integer , the **identity matrix** is an matrix with on the main diagonal and elsewhere and is denoted by

Then , for any matrix .

Inverse of Matrix

For any matrix there is a matrix for which , then is the **inverse** of .

Matrix Solutions of Systems of Equations

For a system of linear equations in variables, , if is an invertible matrix, then unique solution of the system is given by

Determinants of a Matrix

The **determinant** of matrix is denoted and is defined as

Minor

For a square matrix ,the **minor** of an entry is the determinant of the matrix formed by deleting the row and the column of .

Cofactor

For a square matrix , the **cofactor** of of an entry is given by

Where is the minor of

Minor and Cofactors

Note that minors and cofactors are *numbers*. They are not *matrices*.

Determinant of Any Square Matrix

For any square matrix of order , we define the **determinant** of , denoted , as follows. Choose any row or column. Multiply each element in that row or column by its cofactor and add the results. The determinant of a matrix is simply the elements of the matrix. The value of a determinant will be the same no matter which row or column is chosen.

Cramer’s Rule for Systems

The solution of the system of equations

Is given by

Where

and

Cramer’s Rule for Systems

The solution of the system of equations

Were

And

Linear Inequality In Two Variables

A **Linear inequality in two variables** is an inequality that can be written in the form

Where , and are real numbers, and and are not both zero. The symbol may be replaced with , or .

Graph A Linear Inequality in Two Variables

To graph a linear inequality in two variables:

1. Replace the inequality with an equals sign and graph this related equation. If the inequality is or , draw the line dashed. If the inequality is or , draw the line solid
2. The graph consist of a half-plane on one side of the line and, if the line is solid, the line as well. To determine which half-plane to shade, test a point no on the line in the original inequality. If that point is a solution, shade the half-plane containing that point. If not, shade the opposite half-plane.

Linear Programming Procedure

To fine the maximum or minimum value of a linear objective function subject to a set of constraints.

1. Graph the region of feasible solutions
2. Determine the coordinates of the vertices of the region.
3. Evaluate the objective function at each vertex. The largest and smallest of those values are the maximum and minimum values of the function, respectively.

Procedure For Decomposing a Rational Expression into Partial Fractions

Consider any rational expression such that and have no common factor other than or ,

1. If the degree of is greater than or equal to the degree of , divide to express as a quotient remainder and follow stepsto decompose the resulting rational expression.
2. If the degree of is less than the degree of , factor into linear factors of the form and/or quadratic factors of the form . Any quadratic factor must be irreducible, meaning that it cannot be factored into linear factors with rational coefficients.
3. Assign to each linear factor the sum of partial fractions:
4. Assign to each quadratic factor the sum of partial fractions:
5. Apply algebraic methods, to find the constraints in the numerators of the partial fractions.

Parabola

A **Parabola** is the set of all points in a plane equidistant from a fixed line (the **directrix**) and a fixed pint not on the line (the **focus**).

Standard Equation Of A Parabola with vertex at the origin

The standard equation of a parabola vertex and directrix is

The focus is and the **y-axis** is the axis of symmetry.

The standard equation of a parabola with vertex and directrix is

The focus is and the **x-axis** is axis of symmetry.

Standard Equation of A parabola with Vertex and vertical Axis of Symmetry

The standard equation of a parabola with and vertical axis of symmetry is

Where the vertex is , the focus is , and the directrix is

(When , the parabola opens down.)

Standard Equation of A parabola with Vertex and Horizontal Axis of Symmetry

The standard equation of a parabola with and Horizontal axis of symmetry is

Where the vertex is , the focus is , and the directrix is

(When ,the parabola opens to the left.)

Circle

A **circle** is the set of all in a plane that are at a fixed distance from a fixed point (the **center**) in the plane.

Standard Equation Of A Circle

The standard equation of a circle with center and radius is

Ellipse

An **ellipse** is the set of all points in a pane, the sum of whose distances from two fixed points (the **foci**) is constant. The **center** of an ellipse is the midpoint of the segment between the foci.

Standard Equation Of An Ellipse With Center At The Origin

**Major Axis Horizontal**

Vertices:

Y-intercepts:

Foci:, where

**Major Axis Vertical**

Vertices:

Y-intercepts:

Foci:, where

Standard Equation Of An Ellipse With Center At

**Major Axis Horizontal**

Vertices:

Length of minor axis:

Foci: , where

**Major Axis Vertical**

Vertices:

Length of minor axis:

Foci: , where

Hyperbola

A **hyperbola** is the set of all in lane for which the absolute value of the difference of the distances from two fixed points (the **foci**) is constant. The midpoint of the segment between the foci is the **center** of the hyperbola.

Standard Equation Of Hyperbola With Center At The Origin

**Transverse Axis Horizontal**

Vertices:

Foci:, where

**Transverse Axis Vertical**

Vertices:

Foci:, where

Standard Equation Of Hyperbola With Center At The Origin

**Transverse Axis Horizontal**

Vertices:

Asymptotes:

Foci: , where

**Transverse Axis Horizontal**

Vertices:

Asymptotes:

Foci: , where

Rotation of Axes Formulas

If the **x-** and **y-axes** are rotated about the origin through a positive acute angle , then the coordinates and of a point in the and -coordinate systems are related by the following formulas:

Eliminating the xy-Term

To eliminate the **xy-term** from the equation

Select an angle such that

and use the rotation of axes formulas.

Graph of the Equation

The graph of the equation

Is, except in degenerate cases,

1. An ellipse or a circle if
2. A hyperbola if
3. A parabola if

An Alternative Definition Of Conics

Let be a fixed line (the **directrix**), let be a fixed point (the **focus**) not on , and let be a positive constant (the **eccentricity**). A **conic** is the set of all points in the plane such that

Where is the distance from to and is the distance from . The conic is a parabola if an ellipse if , and a hyperbola if .

Eccentricity

For an ellipse and a hyperbola, the **eccentricity** is given by

Where is the distance from the center to a focus and is the distance from the center to a vertex.

Polar Equations of Conics

A polar equation of any of the four forms

Is a conic section. The conic is a parabola if , an ellipse if , and a hyperbola if .

Polar Equations

Vertical directrix unit to the right of the pole (or focus)

Vertical directrix unit to the left of the pole (or focus)

Horizontal directrix units above the pole (or focus)

Horizontal directrix units below the pole (or focus)

Parametric Equations

If and are continuous of on an interval ,then the set of ordered pair such that and is a **plane curve** The equation and are **parametric equations** for the curve. The variable is the **parameter**.

Sequences

An **infinite sequence** is a function having for its domain the set of positive integers,

A **finite sequence** is a function having for its domain a set of positive integers , for some positive integer

Series

Given the infinite sequence

The sum of the terms

Is called an **infinite series**. A **partial sum** is the sum of the first terms:

A partial sum is also called a finite series, or  **partial sum**, and is denoted .

Arithmetic Sequence

A sequence is **arithmetic** if there exists a number , called the **common difference**, such that for any integer

Term of An Arithmetic Sequence

The **term** of an arithmetic sequence is given by

for any integer .

Sum of the First Terms

The sum of the first terms of an arithmetic sequence is given by

Geometric Sequence

A sequence is **geometric** if there is a number , called the **common ratio**, such that

Or

for any integer

Term of An Geometric Sequence

The **term** of a geometric sequence is given by

for any integer .

Sum of the First Terms

The sum of the first terms of an geometric sequence is given by

for any

Limit or Sum of An Infinite Geometric Series

When , the limit or sum of an infinite geometric series is given by

The principle of Mathematical Induction

We can prove an infinite sequence of statements by showing the following

1. Basis step: is true.
2. Induction step: For all natural numbers ,

The Fundamental Counting Principle

Given a combined action, or *event*, in which the first action can be performed ways, the second action can be performed ways, and so on, the total number of ways in which the combined action can be performed is the product

Permutation

A **permutation** of a set of objects is an ordered arrangement of all objects.

The Total Number of Permutations Of Objects

The total number of permutations of objects, denoted as is given by

Factorial Notation

For any natural number

For the number

Generalizing Factorial

For any natural number

Factorial

For any natural numbers and , with

Permutation of Objects Taken At a Time

A **permutation** of a set of objects take at a time is an ordered arrangement of objects from the set.

The Number of Permutations of Objects Taken At A Time

The number of permutations of a set of objects taken at a time, denoted is given by

Number of Distinct Arrangements

The number of distinct arrangements of objects taken at a time, allowing repetition is

Number of Distinguishable Permutations

For a set of objects in which are of one kind are of another kind, , and are of a kind, the number of distinguishable permutations is

Combination; Combination Notation

A **combination** containing objects chosen from a set of objects , is denoted using **combination notation**

Combinations of Objects taken at At A Time

The total number of combinations of taken at at a time, denoted isgiven by

Or

Binomial Coefficient Notation

Subsets of Size and of size

And

The number of subsets of size of a set with objects is the same as the number of subsets of size . The number of combinations of objects taken at a time is the same as the number of combinations of objects taken at a time.

The Binomial Theorem using Pascal’s Triangle

For any binomial and any natural number

Where the numbers are from the row of Pascal’s triangle.

Finding The st Term

The of is

Total Number of Subsets

The total number of subsets of a set with elements is

Principle P (Experimental)

Given an experiment in which observations are made, if a situation, or event, occurs times out of observations, then we say that the **experimental probability** of the event, , is given by

Principle P (Theoretical)

If an event occur ways out of possible equally likely outcomes of a sample space , then the **theoretical probability** of the event , is given by

Probability Properties

1. If an event cannot occur, then
2. If an event is certain to occur, then
3. The probability that an event will occur is a number from to :

Test for Symmetry

1. The graph of an equation in and is symmetric with respect to the **y-axis** when replacing by yields an equivalent equation.
2. The graph of an equation and is symmetric with respect to the **x-axis** when replacing by yields an equivalent equation.
3. The graph of an equation and is symmetric with respect to the **origin** when replacing by and by yields an equivalent equation.

Definition of the Slope of a line

The **slope** of the non-vertical line passing through and is

Slope is not defined for vertical lines.

Slope-Intercept Form of the Equation of a Line

The graph of the linear equation

is a line whose slope is and whose **y-intercept** is

Summary of Equation of Lines

1. General Form:
2. Vertical Line:
3. Horizontal Line:
4. Slope-intercept Form:
5. Point-Slope Form:

Parallel and Perpendicular Lines

1. Two distinct non-vertical lines are **parallel** if and only if their slopes are equal—that is, if and only if
2. Two non-vertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

Definition of a Real-Value Function of a Real Variable

Let and be sets of real numbers. A **real-valued function**  **of a real variable** from to is a correspondence that assigns to each number in exactly one number in .

The **domain** of is the set . The number is the **image** of under and is denoted by , which is called the **value of at** . The **range** of is a subset of and consist of all images of numbers in .

Basic Types of Transformations

Original graph:

Horizontal shift up units to the **right**:

Horizontal shift up units to the **left**:

Vertical shift up units **downward**:

Vertical shift up units **upward**:

**Reflection** (above the **x-axis**):

**Reflection** (above the **y-axis**):

**Reflection** (about the **origin**):

Definition of Composite Function

Let and be functions. The function is the **composite** of with . The domain of is the set of all in the domain of such that is in the domain of .

Test for Even and Odd Functions

The function is **even** when

The function is **odd** when

Definition of the Six Trigonometric Functions

Right triangle definitions, where

Circular function definitions, where is any angle

Trigonometric Identities

**Pythagorean Identities**

Sum and Difference Formula

Even/Odd Identities

Power-Reducing Formulas

Reciprocal Identities

Double-Angle Formulas

Quotient Identities

Slope of Secant Line

Limit of as approaches

Common Types of Behavior Associated with Nonexistence of a Limit

1. approaches a different number from the right side of that it approaches from the left side.
2. increases or decreases without bound as approaches .
3. oscillates between two fixed values as approaches .

Definition of Limit

Let be a function defined on an open interval containing (except possibly at ), and let be a real number. The statement

Means that for each there exists a such that if

Then

Some Basic Limits

Let and be real numbers, and let be a positive integer.

Properties of Limits

Let and be real numbers, and let be a positive integer, and let and be functions with the limits

And

1. Scalar multiple:
2. Sum or difference:
3. Product:
4. Quotient :
5. Power:

Limits of Polynomial and Rational Functions

If is a polynomial function and is a real number, then

If  is a rational function given by and is a real number such that , then

Limit of a Function Involving a Radical

Let be a positive integer. The limit below is valid for all when is odd, and is valid for when is even.

Limit of a Composite Function

If and are function such that and , then

Limits of Trigonometric Functions

Let be a real number in the domain of the given trigonometric function.

Functions That Agree at All but One Point

Let be a real number for all in an open interval containing . If the limit of as approaches exists, then the limit of also exists and

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution.
2. When the limit of as approaches *cannot* be evaluated by direct substitution, try to find a function that agrees with for all other than [ Choose such that the limit of can be evaluated by direct substitution.]
3. Use a *graph* or *table* to reinforce your conclusion.

Squeeze Theorem

If for all in an open interval containing , except possibily at itself, and if

Then exists and is equal to

Two Special Trigonometric Limits

Definition of Continuity

Continuity at a Point

A function is **continuous at**  when these three conditions are met.

1. is defined
2. exists

Continuity on an Open Interval

A function is **continuous on an open interval**  when the function is continuous at each point in the interval. A function that is continuous on the entire real number line is **everywhere continuous**.

Right Sided Limit

Left Sided Limit

One Sided Limit Involving Radicals

Greatest Integer Function

The Existence of a Limit

Let be a function, and let and be real numbers. The limit of as approaches is if and only if

And

Definition of Continuity on a Closed Interval

A function is **continuous on the closed interval**  when is continuous on the open interval and

And

The function is **continuous from the right** at and **continuous from the left** at .

Properties of Continuity

If is a real number and and are continuous at , then the functions listed below are also continuous at .

1. Scalar multiple:
2. Sum or difference:
3. Product:
4. Quotient:

Continuity of a Composite Function

If is continuous at and is continuous at , then the composite function given by is continuous at .

Intermediate Value Theorem

If is continuous on the closed interval , , and is any number between and , then there is at least one number in such that

Definition of Infinite Limits

Let be a function that is defined at every real number in some open interval containing (except possibly at itself). The statement

Means that for each there exists a such that whenever . Similarly, the statement

Means that for each there exists a such that whenever

To define the **infinite limit from the left**, replace by . To define the **infinite limit from the right**, replace by .

Definition of Vertical Asymptote

If approaches infinity (or negative infinity) as approaches from the right or the left, then the line is a **vertical asymptote** of the graph of .

Vertical Asymptotes

Let and be continuous on an open interval containing . If , , and there exists an open interbal containing such that for all in the interval, then graph of the function

Has a vertical asymptote at

Properties of Infinite Limits

Let and be real numbers, and let and be functions such thatand

**Sum or difference:**

**Product:**

**Quotient:**

Similar properties hold for one-sided limits and for functions for which the limit of as approaches is .

Definition of Tangent Line with Slope

If is defined on an open interval containing , and if the limit

Exists, then the line passing through with slope is the **tangent line** to the graph of at the point

Definition of the Derivative of a Function

The **derivative** of at is

Provided the limits exists. For all for which the limits exists, is a function of .

Alternative Form of Derivative

Differentiability Implies Continuity

If is differentiable at , then is continuous at .

The Constant Rule

The derivative of a constant function is . That is, if is a real number, then

The Power Rule

If is a rational number, then the function is differentiable and

For to be differentiable at , must be a number such that is defined on an interval containing .

The Power Rule when

The Constant Multiple Rule

If is a differentiable function and is a real number, then is also differentiable and

The Sum and Difference Rules

The sum (or difference) of two differentiable functions and is itself differentiable. Moreover, the derivative of (or ) is the sum (or difference) of the derivatives of and .

**Sum Rule**

**Difference Rule**

Derivatives of Sine and Cosine Functions

Average Velocity

Velocity Function

Position Function

The Product Rule

The product of two differentiable functions and is itself differentiable. Moreover, the derivative of is the first function times the derivative of the second, plus the second function times the derivative of the first.

The Quotient Rule

The quotient of two differentiable functions and is itself differentiable at all values of for which . Moreover, the derivative of is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Derivatives of Trigonometric Functions

The Chain Rule

If is a differentiable function of and is a differentiable function of , then is a differentiable function of and

Or, equivalently

The General Power Rule

If , where is a differentiable function of and is a rational number, then

Or, equivalently,

Summary of Differentiation Rules

General Differentiation Rules: Let be a real number, let be a rational number, let and be differentiable funtions of , and let be a differentiable function of .

**Constant Rule**

**Constant Multiple Rule**

**Product Rule**

**Chain Rule**

**(Simple) Power Rule**

**Sum or Difference Rule**

**Quotient Rule**

Guidelines for Implicit Differentiation

1. Differentiate both side of the equation with *respect to* .
2. Collect all terms involving on the left side of the equation and move all other terms to the right side of the equation
3. Factor out of the left side of the equation.
4. Solve for .

Guidelines For Solving Related-Rate Problems

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time* .
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Definition of Extrema

Let be defined on an interval containing .

1. is the **minimum of on** when for all in
2. is the **maximum of on** when for all in

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval. Extrema that occur at the endpoints are called **endpoint extrema**.

The Extreme Value Theorem

If is continuous on a closed interval , then has both a minimum and a maximum on the interval.

Definition of Relative Extrema

1. If there is an open interval containing on which is a maximum, then is called a **relative maximum** of , or you can say that has a **relative maximum** at .
2. If there is an open interval containing on which is a minimum, then is called a **relative minimum** of , or you can say that has a **relative minimum** at .

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum**, respectively.

Definition of a Critical Number

Let be defined at . If or if is not differentiable at , then , then , is a **critical number** of .

Relative Extrema Occur Only at Critical Numbers

If has a relative minimum or relative maximum at , then is a critical number of .

Guidelines For Finding Extrema On a Closed Interval

To find the extrema of a continuous function on a closed interval , use these steps

1. Find the critical numbers of in .
2. Evaluate at each critical number in .
3. Evaluate at each endpoint of .
4. The least of these values is the minimum. The greatest is the maximum.

Rolle’s Theorem

Let be continuous on the closed interval and differentiable on the open interval . If , then there is at least one number in such that

The Mean Value Theorem

If is continuous on the closed interval and differentiable on the open interval , then there exists a number in such that

Alternative Form of Mean Value Theorem

Definitions of Increasing and Decreasing Functions

A function is **increasing** on an interval when, for any two numbers and in the interval implies

A function is **decreasing** on an interval when, for any two numbers and in the interval implies

Test for Increasing and Decreasing Functions

Let be a function that is continuous on the closed interval and differentiable on the open interval .

1. If for all in , then is increasing on **.**
2. If for all in , then is decreasing on **.**
3. If for all in , then is constant on **.**

Guidelines For Finding Intervals on Which A Function is Increasing or Decreasing

Let be continuous on the interval . To find the open intervals on which is increasing or decreasing, use the following steps.

1. Locate the critical numbers of in , and use these number to determine test intervals.
2. Determine the sign of at one test value in each of the intervals.
3. Use the previous test for increasing and decreasing functions to determine whether is increasing or decreasing on each interval.

The guidelines are also valid when the interval is replaced by an interval of the form , or .

The First Derivative Test

Let be a critical number of a function that is continuous on an open interval containing . If is differentiable on the interval, except possibly at , then can be classified as follows.

1. If changes from negative to positive at , then has a *relative minimum* at .
2. If changes from positive to negative at , then has a *relative maximum* at .
3. If changes from positive on both sides of or negative on both sides of , then is neither a relative minimum nor a relative maximum.

Definition of Concavity

Let be differentiable on an open interval . The graph of is **concave upward** on when is increasing on the interval and **concave downward** on when is decreasing on the interval.

Test for Concavity

Let be a function whose second derivative exists on an open interval .

1. If for all in , then the graph of is concave upward on
2. If for all in , then the graph of is concave downward on

Definition of Point of Inflection

Let be a function that is continuous on an open interval, and let be a point in the interval. If the graph has a tangent line at the point , then this point is a **point of inflection** of the graph of when the concavity of changes from upward to downward (or downward to upward) at that point.

Points of Inflection

If is a point of inflection of the graph of , then either or does not exist.

Second Derivative Test

Let be a function such that and the second derivative of exists on an open interval containing .

1. If , then has a relative minimum at
2. If , then has a relative maximum at

If , then the test fails. That is, may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Definition of Limits at Infinity

Let be a real number.

1. The statement means that for each there exists an such that whenever
2. The statement means that for each there exists an such that whenever

Definition of a Horizontal Asymptote

The line is a **horizontal asymptote** of the graph of when

Or

Limits at Infinity

If is a positive rational number and is any real number, then

Furthermore, if is defined when , then

Guidelines for Finding Limits at of Rational Functions

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is .
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exists.

Definition of Infinite Limits at Infinity

Let be a function on the interval

1. The means that for each positive number , there is a corresponding number such that whenever .
2. The means that for each negative number , there is a corresponding number such that whenever .

Similar definitions can be given for the statements

And

Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the **x-values** for which and either are zero or do not exists. Use the results to determine relative extrema and points of inflection.

Guidelines For solving Applied Minimum and Maximum Problems

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized.
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by calculus techniques.

Newton’s Method for Approximating the Zeros of a Function

Let , where is differentiable on an open interval containing . Then, to approximate , use these steps.

1. Make an initial estimate that is close to
2. Determine a new approximation
3. When is within the desired accuracy, let serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called **iteration**.

Definition of Differentials

Let represent a function that is differentiable on an open interval containing . The **differential of** (denoted by ) is any non-zero real number. The **differential of**  (denoted by ) is

Differential Formulas

Let and be differentiable functions of .

Constant multiple:

Sum or difference:

Product:

Quotient:

Definition of Antiderivative

A function is an **antiderivative** of on an interval when for all in

Representation of Antiderivatives

If is an antiderivative of on an interval , then is an antiderivative of on the interval if and only if is of the form for all in , where is a constant.

Basic Integration Rules

**Differentiation Formula**

**Integration Formula**

Sigma Notation

The sum of terms is written as

Where is the **index of summation**, is the  **term** of the sum, and the **upper and lower bounds of summation** are and .

Summation Formulas

Limits of the Lower and Upper Sums

Let be continuous and non-negative on the interval . The limits as of both the lower and upper sums exist and are equal to each other. That is,

Where and and are the minimum and maximum values of on the subinterval.

Definition of the Area of a Region in the Plane

Let be continuous and non-negative on the interval . The area of the region bounded by the graph of , the **x-axis**, and the vertical lines and is

Where and

Definition of Riemann Sum

Let be defined on the closed interval , let be a partition of given by

Where is the width of the subinterval

If is *any* point in the subinterval, then the sum

Is called a **Riemann sum** of for the partition .

Regular Partition

General Partition

Definition of Definite Integral

If is defined on the closed interval and the limit of Riemann sums over partitions

Exists (as described above), then is said to be **integrable** on and the limit is denoted by

The limit is called the **definite integral** of from to . The number is the **lower limit** of integration, and the number is the **upper limit** of integration.

Continuity Implies Integrability

If a function is continuous on the closed interval , then is integrable on . That is exists.

The Definite Integral as the Area of a Region

If is continuous and non-negative on the closed interval , then the area of the region bounded by the graph of , the **x-axis**, and the vertical lines and is

Definitions of Two Special Definite Integrals

1. If is defined at , then
2. If is integrable on , then

Additive Interval Property

If is integrable on the three closed intervals determined by , , and , then

Properties of Definite Integrals

If and are integrable on and is a constant, then the functions and are integrable on , and

Preservation of Inequality

1. If integrable and non-negative on the closed interval , then
2. If and are integrable on the closed interval and for every in , then

The Fundamental Theorem of Calculus

If a function is continuous on the closed interval and is an antiderivative on on the interval , then

Guidelines for Using the Fundamental Theorem of Calculus

1. *Provided you can find* an antiderivative of , you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the notation shown below is convenient
3. It is not necessary to include a constant of integration in the antiderivative.

Mean Value Theorem for Integrals

If is continuous on the closed interval , then there exists a number in the closed interval such that

Definition of the Average Value of a Function on an Interval

If is integrable on the closed interval , then the **average value** of on the interval

The Second Fundamental Theorem of Calculus

If is integrable on an open interval containing , then, for every in the interval

The Net Change Theorem

If is the rate of change of a quantity , then the definite integral of from to gives the total change, or **net change**, of on the interval

Antidifferentiation of a Composite Function

Let be a function whose range is an interval , and let be a function that is continuous on . If is differentiable on its domain and is an antiderivative of on , then

Letting gives and

Guidelines for Making a Change of Variables

1. Chose a substitution . Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute .
3. Rewrite the integral in terms of the variable .
4. Find the resulting integral in terms of .
5. Replace by to obtain an antiderivative in terms of .
6. Check your answer by differentiating.

The General Power Rule for Integration

If is a differentiable function of , then

Equivalently, if , then

Change of Variables for Definite Integrals

If the function has a derivative on the closed interval and is continuous on the range of , then

Integration of Even and Odd Functions

Let be integrable on the closed interval

1. If is an *even* function, then
2. If is an *odd* function, then

Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

The domain of the natural logarithmic function is the set of all positive real numbers.

Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties,

1. The domain is and the range is .
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Logarithmic Properties

If and are positive numbers and is rational, then the following properties are true.

Definition of

The letter denotes the positive real number such that

Derivative of the Natural Logarithmic Function

Let be a differentiable function of **.**

Derivative Involving Absolute Value

If  is a differentiable function of such that , then

Log Rule for Integration

Let be a differentiable function of .

Guidelines for Integration

1. Learn a basic list of integration formulas.
2. Find an integration formula that resembles all or part of the integrand and, by trial and error, find a choice of  that will make the integrand conform to the formula.
3. When you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division.
4. If you have access to computer software that will find antiderivatives symbolically use it.
5. Check your result by differentiating to obtain the original integrand.

Integrals of the Six Basic Trigonometric Functions

Definition of Inverse Function

A function is the **inverse function** of the function when

for each in the domain of

And

for each in the domain of .

The function is denoted by (read “ inverse”).

Reflective Property of Inverse Functions

The graph of contains the point if and only if the graph of contains the point .

The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Guidelines for Finding an Inverse Function

1. Determine whether the function has an inverse function.
2. Solve for as a function of .
3. Interchange and . The resulting equation is .
4. Define the domain of as the range of .
5. Verify that and

ontinuity and Differentiability of Inverse Functions

Let be a function whose domain is an interval . If has an inverse function, then the following statements are true.

1. If is continuous on its domain is continuous on its domain.
2. If is increasing on its domain is increasing on its domain.
3. If is decreasing on its domain is decreasing on its domain.
4. If is differentiable on an interval containing and , then is differentiable at .

The Derivative of an Inverse Function

Let be a function that is differentiable on an interval . If has an inverse function , then is differentiable at any for which . Moreover,

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function is called the **natural exponential function** and is denoted by

That is

If and only if

Inverse relationship between natural logarithmic function and natural exponential

And

Operations with Exponential Functions

Let and be any real numbers.

Properties of the Natural Exponential Function

1. The domain of is and the range is .
2. The function is continuous, increasing, and one-to-one on its entire domain.
3. The graph of is concave upward on its entire domain.

The natural exponential function is increasing, and its graph is concave upward.

Derivatives of the Natural Exponential Function

Let be a differentiable function of .

Integration Rules for Exponential Functions

Let be a differentiable function of .

Definition of Exponential Function to Base

If is a positive real number and is any real number, then the **exponential function to the base**  is denoted by and is defined by

If , then is constant function.

Definition of Logarithmic Function to Base

If is a positive real number and is any positive real number. Then the **logarithmic function to the base**  is denoted by and is defined as

Properties of Inverse Functions

1. if and only if
2. , for all

Derivatives for Bass Other than

Let be a positive number , let be a differentiable function of .

The Power Rule for Real Exponents

Let be any real number, and let be a differentiable function of .

A Limit Involving

Summary of Compound Interest Formulas

Let , , , (in decimal form), and .

1. Compounded times per year:
2. Compounded continuously

The Extended Mean Value Theorem

If and are differential on an open interval and continuous on such that for any in , then there exists a point in such that

L’Hpital’s Rule

Let and be functions that are differentiable on an open interval containing, except possibly at itself. Assume that for all in , except possibly at itself. If the limit of as approaches produces the indeterminate form , then

Provided the limit on the right exists (or is infinite). This result also applies when the limit of as approaches produces any one of the indeterminate forms , , , or

Definitions of Inverse Trigonometric Functions

if and only if

Domain:

Range:

if and only if

Domain:

Range:

if and only if

Domain:

Range:

if and only if

Domain:

Range:

if and only if

Domain:

Range:

if and only if

Domain:

Range:

Properties of Inverse Trigonometric Functions

If and , then

And

If , then

And

If and or , then

And

Similar properties hold for other inverse trigonometric functions.

Derivatives of Inverse Trigonometric Functions

Let be a differentiable function of .

Basic Differentiation Rules for Elementary Functions

Integrals Involving Inverse Trigonometric Functions

Let be a differentiable function of , and let

Basic Integration Rules

Definitions of the Hyperbolic Functions

Hyperbolic Identities

Derivatives and Integrals of Hyperbolic Functions

Inverse Hyperbolic Functions

**Domain**

**Domain**

**Domain**

**Domain**

**Domain**

Differentiation and Integration Involving Inverse Hyperbolic Functions

Exponential Growth Decay

If is a differentiable function of such that and for some constant , then

Where is the **initial value** of , and is the **proportionality constant**. **Exponential growth** occurs when , and **exponential decay** occurs when .

Definition of First-Order Linear Differential Equation

A **first-order linear differential equation**  is an equation of the form

Where and are continuous functions of . This first-order linear differential equation is said to be in **standard form**.

Solution of a First-Order Linear Differential Equation

An integrating factor for the first-order linear differential equation

Is . The solution of the differential equation is

Area of a Region between Two Curves

If and are continuous on and for all on , then the area of the region bounded by the graphs and are the vertical lines and is

The Disk Method

To find the volume of a solid of revolution with the **disk method**, use one of the formulas below

**Horizontal Axis of Revolution**

**Vertical Axis of Revolution**

The Washer Method

Volumes of Solids with Known Cross Sections

1. For cross sections of area taken perpendicular to the **x-axis**,
2. For cross sections of area taken perpendicular to the **y-axis**,

The Shell Method

To find the volume of a solid of revolution with the **shell method**, use one of the formulas below

**Horizontal Axis of Revolution**

**Vertical Axis of Revolution**

Definition of Arc Length

Let the function represent a smooth curve on the interval . The **arc length** of between and is

Similarly, for a smooth curve , the **arc length**  of between and is

Definition of Surface of Revolution

When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution**.

Definition of the Area of a Surface Revolution

Let have a continuous derivative on the interval . The area of the surface of revolution formed by revolving the graph of about a horizontal or vertical axis is

Where is the distance between the graph of and the axis of revolution. If is on the interval , then the surface area is

Where is the distance between the graph of and the axis of revolution.

Definition of Work Done by a Constant Force

If an object is moved a distance in the direction of an applied constant force , then the **work**  done by the force is defined as .

Definition of Work Done by a Variable Force

If an object is moved along a straight line by a continuously varying force , then the **work** done by the force as the object is moved from

To

Is given by

Hooke’s Law

The force required to compress or stretch a spring (within its elastic limits) is proportional to the distance that the spring is compressed or stretched from its original length. That is

Newton’s Law of Universal Gravitation

The force of attraction between two particles of masses and is proportional to the product of the masses and inversely proportional to the square of the distance between the two particles. That is

Coulomb’s Law

The force between two charges and in a vacuum is proportional to the product of the changes and inversely proportional to the square of the distance between the two charges. That is,

When and are given electrostatic units and in centimeters, will be in dynes for a value of .

Force

Moment

The **moment of about the point is**

Moment and Center of Mass: One-Dimensional System

Let the point masses be located at .

1. The **moment about the origin** is
2. The **center of mass** is

Where is the **total mass** of the system.

Moment and Center of Mass: Two-Dimensional System

Let the point masses be located at

1. The **moment about the y-axis** is
2. The **moment about the x-axis**
3. The **center of mass**  (or **center of gravity**) is

And

Where

is the **total mass** of the system.

Moments and Center of Mass of a Planar Lamina

Let and be continuous functions such that on , and consider the planar lamina of uniform density bounded by the graphs of , and .

1. The **moments about the x- and y-axis** are
2. The **center of mass**  is given by and , where is the mass of the lamina.

The Theorem of Pappus

Let be in a region in a plane and let be a line in the same plane such that does not intersect the interior of . If is the distance between the centroid of and the line, then the volume of the solid of revolution formed by revolving about the line is

Where is the area of . (Note that is the distance traveled by the centroid as the region is revolved about the line.)

Definition of Fluid Pressure

The **pressure**  on an object at depth in a liquid is

Where is the weight-density of the liquid per unit of volume.

Definition of Force Exerted by a Fluid

The **force exerted by a fluid** of constant weight-density (per unit of volume) against a submerged vertical plane region from to is

Where is the depth of the fluid at and is the horizontal length of the region at .

Procedures for Fitting Integrands to Basic Integration Rules

**Technique**

1. Expand (numerator)
2. Separate Numerator
3. Complete the Square
4. Divide improper rational function
5. Add and Subtract terms in numerator
6. Use trigonometric identities
7. Multiply and Divide by Pythagorean conjugate

Integration by Parts

If and are functions of and have continuous derivative, then

Guidelines for Integration by Parts

1. Try letting be the most complicated portion of the integrand that fits a basic integration rule. Then will be the remaining factor(s) of the integrand.
2. Try letting be the portion of the integrand whose derivative is a function simpler than . Then will be the remaining factor(s) of the integrand.

Note that always includes the of the original integrand.

Summary: Common Integrals using Integration By Parts

1. For integrals of the form

Let and let .

1. For integrals of the form

Let and let

1. For integrals of the form

Let or and let

Guidelines for Evaluating Integrals Involving Powers of Sine and Cosine

1. When the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then expand and integrate.
2. When the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then expand and integrate.
3. When the powers of both the sine and cosine are even and non-negative, make repeated use of the formulas

And

To convert the integrand to odd powers of the cosine. Then proceed as in the second guideline.

Wallis’s Formulas

1. If is odd , then
2. If is odd , then

The formulas are also valid when is replaced by

Guidelines for Evaluating Integrals Involving Powers of Secant and Tangent

1. When the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.
2. When the power of the secant is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.
3. When there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.
4. When the integral of the form

When is odd and positive, use integration by parts

1. When the first four guidelines do not apply, try converting to sines and cosines.

Integrals Involving Sine-Cosine Products

Trigonometric Substitution

1. For integrals involving , let

Then , where

1. For integrals involving , let

Then , where

1. For integrals involving , let

Then

Special Integration Formulas

Decomposition of into Partial Fractions

1. **Divide when improper:** When is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

Where the degree of is less than the degree of . Then apply Steps 2, 3, and 4 to the proper rational express

1. **Factor denominator:** Completely factor the denominator into factors of the form

and

Where is irreducible.

1. **Linear factors:** For each factor of the form , the partial fraction decomposition must include the following sum of fractions
2. **Quadratic factors:** For each factor of the form , the partial fraction decomposition must include the following sum of fractions

Guidelines for Solving the Basic Equation

**Linear Factors**

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determines in the first guideline to rewrite the basic equation. Then substitute other convenient values of and solve for the remaining coefficients.

**Quadratic Factors**

1. Expand the basic equation.
2. Collect terms according to powers of .
3. Equate the coefficients of like power to obtain a system of linear equations involving , and so on.
4. Solve the system of linear equations.

The Trapezoidal Rule

Let be continuous on . The Trapezoidal Rule for approximating is

Moreover, as , the right-hand side approaches .

Integration of

If , then

Simpson’s Rule

Let be continuous on and let be an even integer. Simpson’s Rule for approximating is

Moreover, as , the right-hand side approaches .

Errors in the Trapezoidal Rule and Simpson’s Rule

If has a continuous second derivative on , then the error in approximating by the Trapezoidal Rule is

Moreover, if as a continuous fourth derivative on , then the error in approximating by Simpson’s Rule is

Substitution for Rational Functions of Sine and Cosine

For integrals involving rational functions of sine and cosine, the substitution

Yields

Definition of Improper Integrals with Integration Limits

1. If is continuous on the interval , then
2. If is continuous on the interval , then
3. 3 If is continuous on the interval , then

Where is any real number.

Definition of Improper Integrals with Infinite Discontinuities

1. If is continuous on the interval and has infinite discontinuity at , then
2. If is continuous on the interval and has infinite discontinuity at , then
3. If is continuous on the interval , except for some in at which has an infinite discontinuity, then

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

A Special Improper Integral

Definition of the Limit of a Sequence

Let be a real number. The **limit** of a sequence is , written as

If for each , there exists such that whenever . If the limit of a sequence exists, then the sequence **converges** to . If the limit of a sequence does not exist, then the sequence **diverges**.

Limit of a Sequence

Let be a real number. Let be a function of a real variable such that

If is a sequence such that for every positive integer , then

Properties of Limits of Sequences

Let and

**Scalar multiple:** is any real number

**Sum or Difference**

**Product:**

**Quotient:** and

Squeeze Theorem for Sequences

If and there exists an integer such that for all , then

Absolute Value Theorem

For the sequence if

Then

Definition of Monotonic Sequence

A sequence is **monotonic** when its terms are non-decreasing

Or when its terms are non-increasing

Definition of Bounded Sequence

1. A sequence is **bounded above** when there is a real number such that for all . The is called an **upper bound** of the sequence.
2. A sequence is **bounded below** when there is a real number such that for all . The number is called a **lower bound** of the sequence.
3. A sequence is **bounded** when it is bounded above and bounded below.

Bounded Monotonic Sequences

If a sequence is bounded and monotonic, then it converges

Infinite Series

Definitions of Convergent and Divergent Series

For the infinite series , the **nth partial sum** is

If the sequence of partial sums converges to , then the series **converges**. The limit is called the **sum of the series**.

If diverges, then the series **diverges**.

Telescoping Series

Geometric Series

Convergence of a Geometric Series

A geometric series with ratio diverges when . If , then the series converges to the sum

Properties of Infinite Series

Let and be convergent series, and let and be real numbers. If and , then the following series converge to indicated sums.

Limit of the term of a Convergent Series

If converges, then

term of a Divergent Series

If then diverges.

The Integral Test

If is positive, continuous, and decreasing for and , then

And

Either both converge or both diverge.

P-series

Harmonic Series

Convergence of p-series

The p-series

Converges for and diverges for

Direct Comparison Test

Let for all

1. If converges, then converges
2. If diverges, then diverges

Limit Comparison Test

If an

Where is *finite and positive*, then

And

Either both converge or both diverge.

Alternating Series Test

Let . The alternating series

And

Converge when these two conditions are met

1. for all

Alternating Series Remainder

If a convergent alternating series satisfies the condition , then the absolute value of the remainder involved in approximating the sum by is less than (or equal to) the first neglected term. That is,

Absolute Convergence

If the series converges, then the series also converges.

Definitions of Absolute and Conditional Convergence

1. The series is **absolutely convergent** when converges.
2. The series is **conditionally convergent** when converges but diverges.

Ratio Test

Let be a series with non-zero terms

1. The series converges absolutely when
2. The series diverges when or
3. The Ratio Test is inconclusive when

Root Test

1. The series converges absolutely when
2. The series diverges when or
3. The Root Test is inconclusive when

Guidelines for Testing a Series for Convergence or Divergence

1. Does the root term approach ? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

Definitions of Taylor Polynomial and Maclaurin Polynomial

If has derivatives at , then the polynomial

Is called the **Taylor polynomial for** at . If

Is also called the **Taylor polynomial for** .

Taylor Polynomial Remainder

Error in Taylor Polynomial

Taylor’s Theorem

If a function is differentiable through order in an interval containing , then, for each in , there exists between and such that

Where

Definition of Power Series

If is a variable, then an infinite series of the form

Is called a **power series**. More generally, an infinite series of the form

Is called a **power series centered at** , where is a constant.

Convergence of a Power Series

For a power series centered at , precisely one of the following is true,

1. The series converges only at .
2. There exists a real number such that the series converges absolutely for

And diverges for

1. The series converges absolutely for all

The number is the **radius of convergence** of the power series. If the series converges only at , then the radius of convergence . If the series converges at all , then the radius convergence is . The set of all values of for which the power series converges is the **interval of convergence** of the power series.

Properties of Function Defined by Power Series

If the function

Has a radius of convergence of , then one the interval

is a differentiable (and therefore continuous). Moreover, the derivative and antiderivative of are as follows.

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Operations with Power Series

Let and

The Form of a Convergent Power Series

If is represented by a power series for all in an open interval containing , then

And

Definition of Taylor and Maclaurin Series

If a function has derivative of all orders at , then the series

Is called the **Taylor series for at** , then the series is the **Maclaurin series for** .

Convergence of Taylor Series

If for all in the interval , then the Taylor series for converges and equals

Guidelines for Finding a Taylor Series

1. Differentiate with respect to several times and evaluate each derivative at .

Try to recognize a patter in these numbers

1. Use the sequence developed in the first step to form the Taylor coefficients and determine the interval of convergence for the resulting power series
2. Within this interval of convergence, determine whether the series converges to

General Second-Degree Equation

Standard Equation of a circle

Standard Equation of a Parabola

The **standard form** of the equation of a parabola with vertex and directrix is

**Vertical axis**

For directrix , the equation is

**Horizontal axis**

The focus lies on the axis units (*directed distance*) from the vertex. The coordinates of the focus are as follows

**Vertical axis**

**Horizontal axis**

Reflective Property of a Parabola

Let be on a point on a parabola. The tangent line to the parabola at point makes equal angles with the following two lines.

1. The line passing through and the focus.
2. The line passing through parallel to the axis of the parabola.

Standard Equation of an Ellipse

The **standard form** of the equation of an ellipse with center and major and minor axes of lengths and , respectfully, where , is

**Major Axis Horizontal**

Or

**Major Axis Vertical**

The foci lie on the major axis, units from the center, with

Reflective Property of an Ellipse

Let be a point on an ellipse. The tangent line to the ellipse at point makes equal angles with the line through and the foci.

Definition of Eccentricity of an Ellipse

Standard Equation of a Hyperbola

The **standard form** of the equation of a hyperbola with center at is

**Transverse Axis Horizontal**

Or

**Transverse Axis Vertical**

The vertices are units from the center, and the foci are units from the center, where

Asymptotes of a Hyperbola

For a *horizontal* transverse axis, the equation of the asymptotes are

And

For a *vertical* transverse axis, the equations of the asymptotes are

And

Definition of Eccentricity of a Hyperbola

The **eccentricity**  of a hyperbola is given the ratio

Definition of a Plane Curve

If and are continuous functions of on an interval , then the equations

And

Are **parametric equations** and is the **parameter**. The set of points obtained as varies on the interval is the **graph** of the parametric equations. Taken together, the parametric equations and the graph are a **plane curve**, denoted by .

Definition of a Smooth Curve

A curve represented byand on an interval and not simultaneously , except possibly at the endpoints of . The curve is called **piecewise smooth** when it is smooth on each subinterval of some partition of .

Parametric Form of the Derivative

If a smooth curve is given by the equations

And

Then the slope of at is

Arc Length in Parametric Form

If a smooth curve is given by and such that does not intersect itself on the interval (except possibly at the endpoints), then the arc length of over the interval is given by

Area of a Surface of Revolution

If a smooth curve is given by and such that does not intersect itself on the interval , then the area of the surface of revolution formed by revolving about the coordinate axes is given by the following.

**Revolution about the x-axis:**

**Revolution about the y-axis:**

Coordinate Conversion

The polar coordinates of a point are related to the rectangular coordinates of the point as follows

**Polar to Rectangular**

**Rectangular to Polar**

Slope in Polar Form

If is a differentiable function of , then the *slope* of the tangent line to the graph of at the point is

Provided that at

Tangent Lines at the Pole

If and , then the line is tangent at the pole to the graph of .

Area in Polar Coordinates

If is continuous and non-negative on the interval , then the area of the region bounded by the graph of between the radial lines and is

Arc Length of a Polar Curve

Let be a function whose derivative is continuous on an interval . The length of the graph from to is

Area of a Surface of Revolution

Let be a function whose derivative is continuous on an interval . The area of the surface formed by revolving the graph of from to about the indicated line is as follows

**About the polar axis**

**About the line**

Classification of Conics by Eccentricity

Let be a fixed point and let be a fixed line in the plane. Let be another point in the plane and let be the ratio of the distance between and to the distance between and . The collection of all points with a given eccentricity is a conic.

1. The conic is an ellipse for
2. The conic is a parabola for
3. The conic is a hyperbola for

Polar Equations of Conics

The graph of a polar equation of the form

Or

Is a conic, where is the eccentricity and is the distance between the focus at the pole and its corresponding directrix.

Determining a Conic from its Equation

**Ellipse**

**Hyperbola**

Definition of Component Form of a Vector in the Plane

If is a vector in the plane whose initial point is the origin and whose terminal point is , then the **component form of**  is . The coordinates and are called the **components of** . If both the initial point and the terminal point lie at the origin of is called the **zero vector** and is denoted by .

Length of a Vector

Definitions of Vector Addition and Scalar Multiplication

Let and be vectors and let be a scalar.

1. The **vector sum** of and is the vector
2. The **vector multiple** of and is the vector
3. The **negative** of is the vector
4. The **difference** of and is

Properties of Vector Operations

Let and be vectors in the plane, and let and be scalars

## Communicative Properties

## Associative Property

## Additive Identity Property

## Distributive Property

Length of a Scalar Multiple

Let be a vector and let be a scalar. Then

is the absolute value of .

Unit Vector in the Direction of

If is a non-zero vector in the plane, then the vector

Has length and the same direction as .

Standard Unit Vector

And

Equation of a Sphere

Midpoint Formula

Vectors in Space

Let and be vectors in space and let be a scalar.

1. *Equality of Vectors*: if and only if

and

1. *Component Form*: If is represented by the directed line segment from to , then
2. *Length*:
3. *Unit Vector in the Direction of* :
4. *Vector Addition*:
5. *Scalar Multiplication*:

Definition of Parallel Vectors

Two non-zero vectors and are **parallel** when there is some scalar such that .

Definition of Dot Product

The **dot product** of and

The **dot product** of and is

Properties of the Dot Product

Let and be vectors in the plane, and let be a scalar

## Communicative Properties

## Distributive Property

## Associative Property

Angle Between Two Vectors

If is the angle between two non-zero vectors and , where , then

Definition of Orthogonal Vectors

The vectors and are orthogonal when

Alternative form of Dot Product

Let be non-zero vectors. Moreover, let

Where is parallel to and is orthogonal,

1. is called the **projection of onto**  or the **vector component of along** , and is denoted by
2. is called the **vector component of orthogonal to**

Projection Using the Dot Product

If and are non-zero vectors, then the projection of onto is

Definition of Work

The work done by a constant force as its point of application moves along the vector is one of the following

**Projection form**

**Dot Product form**

Definition of Cross Product of Two Vectors in Space

Let

And

Be vectors in space. The **cross product** of and is the vector

Algebraic Properties of the Cross Product

Let and be vectors in space, and let be a scalar.

Geometric Properties of the Cross Product

Let be non-zero vectors in space, and let **θ** the angle and t .

1. is orthogonal to both t and .
2. if and only if and are scalar multiples of each other.
3. having and as adjacent sides.

The Triple Scalar Product

For and , the triple scalar product is

Geometric Property of the Triple Scalar Product

The volume of a parallelepiped with vectors and as adjacent edges is

Parametric Equations of a Line in Space

A line parallel to the vector and passing through the point is represented by the **parametric equations**

Symmetric Equations

Standard Equation of a Plane in Space

The plane containing the point and having normal vector

Can be represented by the **standard form** of the equation of a plane

General form of Equation of plane

Angle between two planes

Distance Between a Point and a Plane

The distance between a plane and a point (not in the plane) is

Where is appoint in the plane and is normal to the plane

Distance Between a Point and a Plane

Distance Between a Point and a in Space

The distance between a point and a line in space is

Where is a direction vector for the line and is a point on the line.

Definition of a Cylinder

Let be a curve in a plane and let be a line not in a parallel plane. The set of all lines parallel to and intersecting is a **cylinder**. The curve is the **generating curve** (or **directrix**) of the cylinder, and the parallel lines are **rulings**.

Equations of Cylinders

The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variable corresponding to the other two axes.

Quadric Surface

The equation of a **quadric surface** in space is a second-degree equation in three variables. The **general form** of the equation is

There are six basic types of quadric surfaces: **ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid,** and **hyperbolic paraboloid**.

Surface of Revolution

If the graph of a radius function is revolved about one of the coordinate axes, then the equation of the resulting surface of revolution has one of the forms.

1. Revolved about the **x-axis**:
2. Revolved about the **y-axis**:
3. Revolved about the **z-axis**:

The Cylindrical Coordinate System

In a **cylindrical coordinate system**, a point in space is represented by an ordered triple .

1. is a polar representation of the projection of in the **xy-plane**.
2. is the directed distance from to .

Cylindrical to Rectangular

Rectangular to Cylindrical

The Spherical Coordinate System

In a **spherical coordinate system**, a point in space is represented by an ordered triple , where is the lowercase Greek letter rho and is the lowercase Greek letter phi

1. is the distance between  **and the origin** .
2. is the same angle used in cylindrical coordinated
3. is the angle *between* the positive **z-axis** and the line segment

Note that the first and third coordinates, and are non-negative.

Spherical to Rectangular

Rectangular to spherical

Spherical to cylindrical

Cylindrical to Spherical

Definition of Vector-Values Function

A function of the form

**Plane**

Or

**Space**

Is a **vector-valued function**, where the **component functions** and are real-valued functions of the parameter . Vector-valued functions are sometimes denoted as

**Plane**

Or

Definition of the Limit of a Vector-Valued Function

1. If is a vector-valued function such that , then

**Plane**

Provided and have limits as

1. If is a vector-valued function such that , then

Provided and have limits as

Definition of Continuity of a Vector-Valued Function

A vector-value d function is **continuous at the point** given by when the limit exists as and

A vector-valued function is **continuous on an interval**  when it is continuous at every point in the interval.

Definition of the Derivative of a Vector-Valued Function

The **derivative of a vector-valued function**  is

For all for which the limit exists. If exists, then is **differentiable at** . If exists for all in an interval , then is **differentiable on the interval** . Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

Differentiation of Vector-Valued Functions

1. If , where and are differentiable functions of , then

**Plane**

1. If , where and are differentiable functions of , then

**Space**

Properties of the Derivative

Let and be differentiable vector-valued functions of , let be a differentiable real-valued function of , and let be a scalar.

If , then

Note Property 5 applies only to three-dimensional vector-valued functions because the cross product is not defined for two-dimensional vectors.

Definition of Integration of Vector-Valued Functions

1. If , where and are continuous on , then the **indefinite integral** (**antiderivative**) of is

**Plane**

And its **definite integral** over the interval over the interval is

1. If , where If and are continuous on , then the **indefinite integral** (**antiderivative**) of is

**Space**

And its **definite integral** over the interval over the interval is

Definitions of Velocity and Acceleration

If and are twice-differentiable functions of , and is a vector-valued function given by , then the velocity vector, acceleration vector, and speed at time are as follows

Position Vector

Position Vector for a Projectile

Neglecting air resistance, the path of a projectile launched from an initial height with initial speed and angle of elevation is described by the vector function

Where is the acceleration due to gravity.

Definition of Unit Tangent Vector

Let be a smooth curve represented by on an open interval . The **unit tangent vector**  at is defined as

Definition of Principal Unit Normal Vector

Let be a smooth curve represented by on an open interval . If , then the **principal normal vector** at is defined as

Acceleration Vector

If is the position vector for a smooth curve and exists, then the acceleration vector lies in the plane determined by and

Tangential and Normal Components of Acceleration

If is the position vector for a smooth curve [for which exists], then the tangential and normal components of acceleration are as follows

Note that . The normal component of acceleration is also called the **centripetal component of acceleration**.

Arc Length of a Space Curve

If is a smooth curve given by on an interval , then the arc length of on the interval is

Definition of Arc Length Function

Let be a smooth curve given by defined on the closed interval . For , the **arc length function** is

The arc length is called the **arc length parameter**.

Arc Length Parameter

If is a smooth cure given by

**Plane Curve**

Or

**Space Curve**

Where is the arc length parameter, then

Moreover, if is *any* parameter for the vector-valued function such that , then must be the arc length parameter.

Definition of Curvature

Let be a smooth curve (in the plane *or* in space) given , where is the arc length parameter. The **curvature**  at is

Formulas for Curvature

If is a smooth curve given by , then the curvature of at is

Curvature in Rectangular Coordinates

If is the graph of a twice-differentiable function given by, then the curvature at the point is

Acceleration, Speed, and Curvature

If is the position vector for a smooth curve , then the acceleration vector is given by

Where is the curvature of and is the speed.

Summary of Velocity, Acceleration, and Curvature

Unless noted otherwise, let be a curvature (in the plane or in space) given by the position vector

**Curve in the Plane**

Or

**Curve in Space**

Where and are twice-differentiable functions of

Velocity vector, speed, and acceleration vector

**Velocity Vector**

**Speed**

**Acceleration Vector**

Components of Acceleration

Formulas for Curvature in the plane

Formulas for Curvature in the Plane or in Space

Cross product formulas apply only to curves in space.

Definition of a Function of Two Variables

Let be a set of ordered pairs of real numbers. If to each ordered pair in there corresponds a unique real number , then is a **function of**  **and** . The set is the **domain** of , and the corresponding set of values for is the **range** of . For the function

and are called the **independent variables** and is called the **dependent variable**.

Operations on Functions with Several Variables

**Sum or Difference**

**Product**

**Quotient**

Composition on Functions with Several Variables

Open Disk

Closed Disk

Definition of he Limit of a Function of wo Variables

Let be a function of two variables defined, except possibly and let be a real number. Then

If for each there corresponds a such that

Whenever

Definition of Continuity of a Function of Two Variables

A function of two variables is **continuous at a point** in an open region if is defined and is equal to the limit of as approaches . That is

The function is **continuous in the open region R** if it is continuous at every point in R.

Continuous Function of Two Variables

If is a real number and are continuous at , then the following functions are also continuous at .

**Scalar Multiple**:

**Sum or Difference**:

**Product**:

**Quotient:**

Continuity of a Composite Function

If is continuous at and is continuous at , then the composite function given by is continuous at That is,

Open Sphere

Definition of Continuity of a Function of Three Variables

A function of three variables is **continuous at a point**  is defined and is equal to the limit of as approaches . That is

The function is **continuous in the open region**  if it is continuous at every point in .

Definition of Partial Derivative of a Function of Two Variables

If , then the first **partial derivatives** of with respect to and are the functions and defined by

**Partial Derivative with respect to x**

**Partial Derivative with respect to y**

Provided the limits exists.

Notation for First Partial Derivatives

For , the partial derivatives and are denoted by

**Partial derivative with respect to x**

**and**

**Partial Derivative with respect to y**

The first partials evaluated at the point are denoted by

and

Differentiate twice with respect to x

Differentiate twice with respect to y

Differentiate first with respect to x and then with respect to y

Differentiate first with respect to y and then with respect to x

Equality of Mixed Partial Derivatives

If is a function of and such that and are continuous on an open disk , then for every in ,

Increment of

Definition of Total Differential

If and and are increments of and , then the **differentials** of the independent variables and are

And

And the **total differential** of the dependent variable is

Definition of Differentiability

A function given by is **differentiable** at if can be written in the form

Where both and as

The function is **differentiable in a region**  if it is differentiable at each point in .

Sufficient Condition for Differentiability

If is a function of and , where and are continuous in an open region , then is differentiable on .

Differentiability Implies Continuity

If a function of and is differentiable at , the it is continuous at .

Chain Rule: One Independent Variable

Let , where is a differentiable function of and . If and , where and are differentiable functions of , then is a differentiable function of , and

Chain Rule: Two Independent Variables

Let , where is a differentiable function of and . If and , such that the first partials , and all exists, then and exist and are given by

And

Chain Rule: Implicit Differentiation

If the equation defines as a differentiable function of , then

If the equation defines implicitly as a differentiable function of and , then

And

Definition of Directional Derivative

Let be a function of two variables and and let be a unit vector. Then the **directional derivative of**  in the direction of , denoted by is

Provided the limit exists.

Directional Derivative

If is a differentiable function of and , then the directional derivative of in the direction of the unit vector is

Definition of Gradient of a Function of Two Variables

Let be a function of and such that and exist. Then the **gradient of** , denoted by , is the vector

(The symbol is read as ‘del.”) Another notation for the gradient is given by **grad** . Note that for each , the gradient is a vector in the plane (not a vector in space).

Alternative Form of the Directional Derivative

If is a differentiable function of and , then the directional derivative of in the direction of the unit vector is

Properties of the Gradient

Let be differentiable at the point .

1. If , then for all .
2. The direction of *maximum increase* of is given by . The maximum value of is
3. The direction of *minimum increase* of is given by . The maximum value of is

Gradient is Normal to Level Curves

If is differentiable at and , then is normal to the level curve through .

Directional Derivative and Gradient for Three Variables

Let be a function of and with continuous first partial derivatives. The **directional derivative of**  in the direction of a unit vector

Is given by

The **gradient of**  is defined as

Properties of the gradient are as follows.

1. If then for all
2. The direction of *maximum increase* of is given by . The maximum value of is
3. The direction of *minimum increase* of is given by . The maximum value of is

Definitions of Tangent Plane and Normal Line

Let be differentiable at the point on the surface given by such that

1. The plane through that is normal to is called the **tangent plane to**  at .
2. The line through having the direction of is called the **normal line to**  at

Equation of Tangent Plane

If is differentiable at , then an equation of the tangent plane to the surface given at is

Angle of inclination of a plane

Gradient is Normal to Level Surfaces

If is differentiable at and

Then is normal to the level surface through .

Extreme Value Theorem

Let be a continuous function of two variables and defined on a closed bounded region in the **xy-plane**.

1. There is least one point in at which takes on minimum value.
2. There is least one point in at which takes on maximum value.

Definition of Relative Extrema

Let be a function defined on a region containing .

1. The function has a **relative minimum** at if

For all in *open disk* containing **.**

1. The function has a **relative maximum** at if

For all in *open disk* containing **.**

Definition of Critical Point

Let be defined on an open region containing . The point is a **critical point** of if one of the following is true.

1. and
2. or does not exist

Relative Extrema Occur Only at Critical Points

If has a relative extremum at on an open region , then is a critical point of .

Second Partials Test

Let have continuous second partial derivatives on an open region containing a point for which

And

To test for relative extrema of , consider the quantity

1. If and , then has a **relative minimum** at
2. If and , then has a **relative maximum** at
3. If , then is a **saddle point**.
4. The test is inconclusive if

Sum of the Squared errors

Least Squares Regression Line

The **least squares regression line** for is given by , where

And

Lagrange’s Theorem

Let and have a continuous first partial derivative such that has an extremum at point on the smooth constraint curve If

Method of Lagrange Multipliers

Let and satisfy the hypothesis of LaGrange’s Theorem, and let have a minimum or maximum subject to the constraint . To find the minimum or maximum of , use these steps.

1. Simultaneously solve the equations and by solving the following system of equations.
2. Evaluate at each solution point obtained in the first step. The greatest value yields the maximum of subject to the constraints , and the least value yields the minimum of subject to the constraints

Area of a Region in the Plane

1. If is defined by and ,where and are continuous on , then the area of is

**Vertically simple**

1. If is defined by and ,where and are continuous on , then the area of is

**Horizontally simple**

Definition of Double Integral

If is defined on a closed, bounded region in the **xy-plane**, then the **double integral of over**  is

Provided the limit exists. If the limit exists, then is **integrable** over

Volume of a Solid Region

If is integrable over a plane region and in , then the volume of the solid region that lies above and below the graph of is

Properties of Double Integrals

Let and be continuous over a closed, bounded plane region , and let be a constant.

Where R is the union of two non-overlapping sub-regions and .

Fubini’s Theorem

Let be continuous on a plane region .

1. If is defined by and , where and are continuous on , then
2. If is defined by and , where and are continuous on , then

Definition of the Average Value of a Function Over a Region

If is integrable over the plane region , then the **average value** of over is

Where is the area of .

Polar sectors

Change of Variables to Polar Form

Let be a plane region consisting of all points satisfying the conditions , where . If and are continuous on and is continuous on , then

Definition of Mass of a Planar Lamina of Variable Density

If is a continuous density function on the lamina corresponding to a plane region , then the mass of the lamina is given by

**Variable Density**

Moments and Center of Mass of a Variable Density Planar Lamina

let be a continuous density function on the planar lamina If . The **moments of mass** with respect to the **x-** and **y-axis** are

And

If is the mass of the lamina, then the **center of mass** is

If represents a simple plane region rather than a lamina, then the point is called the **centroid** of the region.

Definition of Surface Area

If and its first partial derivatives are continuous on the closed region in the **xy-plane**, then the **area of the surface**  over is defined as

Definition of Triple Integral

If is continuous over a bounded solid region , then the **triple integral of over**  is defined as

Provided the limit exists. The **volume** of the solid region is given by

Evaluation by Iterated Integrals

Let be continuous on a solid region defined by

Where and are continuous functions. Then

Triple Integral in Cylindrical form

Triple Integrals in Spherical Coordinates

Definition of the Jacobian

If and, then the **Jacobian** of and with respect to and , denoted by , is

Change of Variables for Double Integrals

Let be a vertically or horizontally simple region in the **xy-plane**, and let be a vertically or horizontally simple region in the **uv-plane**. Let from to be given by , where and have continuous first partial derivatives. Assume that is one-to-one except possibly on the boundary of . If is continuous on **, and**  is non-zero on , then

Definition of Vector Field

A **vector field over a plane region**  is a function that assigns a vector to each point in .

A **vector field over a solid region Q in space** is a function to each point in **.**

Definition of Inverse Square Field

Let be a position vector. The vector field is an **inverse square field** if

Where is a real number and

Is a unit vector in the direction of .

Definition of Conservative Vector Field

A vector field is called **conservative** when there exists a differentiable function such that . The function is called the **potential function** for .

Test for Conservative Vector Field in the Plane

Let and have continuous first partial derivatives on an open disk . The vector field is conservative if and only if

Definition of Curl of a Vector Field

The curl of is

Curl

If curl , then is said to be **irrotational**.

Test for Conservation Vector Field in Space

Suppose that , and have continuous first partial derivatives in an open sphere in space. The vector field

Is conservative if and only if

Curl .

That is, is conservative if and only if

And

Definition of Divergence of a Vector Field

The **divergence** of is

**Plane**

The **divergence** of is

**Space**

If , then is said to be **divergence free**.

Divergence and Curl

If is a vector field and and have continuous second partial derivatives, then

Definition of Line Integral

If is defined in a region containing a smooth curve of finite length, then the **line integral of**  **along**  is given by

**Plane**

Or

**Space**

Provided the limit exists

Evaluation of a Line Integral as a Definite Integral

Let be continuous in a region containing a smooth curve . If is given by , where , then

If  **is given by** , where , then

Definition of the Line Integral of a Vector Field

Let be a continuous vector field defined on a smooth curve given by

The **line integral** of on is given by

Fundamental Theorem of Line Integrals

Let be a piecewise smooth curve lying in an open region and given by

If is conservative in , and and are continuous in , then

Where is a potential function of . That is .

Independence of Path and Conservative Vector FieldsI

Ifis continuous on an open connected region, then the line integral

Is independent of path if and only if is conservative.

Equivalent Conditions

Let have continuous first partial derivatives in an open connected region , and let be a piecewise smooth curve in . The conditions listed below are equivalent.

1. is conservative. That is, for some function .
2. is independent of path.

for *every closed* curve in

Green’s Theorem

Let be a simply connected region with a piecewise smooth boundary , oriented counterclockwise (that is, is traversed *once* so that the region always lies to the left). If and have continuous first partial derivatives in an open region containing , then

Line Integral for Area

If is a plane region bounded by a piecewise smooth simple closed curve , oriented counterclockwise, then the area of is given by

Definition of Parametric Surface

Let , and be functions of and that are continuous on a domain in the **uv-plane**. The set of points given by

**Parametric Surface**

Is called a **parametric surface**. The equations

**Parametric equations**

Are the **parametric equations** for the surface.

Normal Vector to a Smooth Parametric Surface

Let be a smooth parametric surface

Defined over an open region in the **uv-plane**. Let be a point in . A normal vector at the point

Is given by

Area of a Parametric Surface

Let be a smooth parametric surface

Defined over an open region in the **uv-plane**. If each point on the surface corresponds to exactly one point in the domain , then the **surface area** of is given by

Where

And

Evaluating a Surface Integral

Let be a surface given by and let be its projection onto the **xy-plane**. If and are continuous on and is continuous on , then the surface integral of over is

Definition of Flux Integral

Let , where , and have continuous first partial derivatives on the surface oriented by a unit normal vector . The **flux integral of across**  is given by

Evaluating a Flux Integral

Let be an oriented surface given by and let be its projection onto the **xy-plane**.

**Oriented upward**

**Oriented downward**

For the first integral, the surface is oriented upward, and for the second the integral, the surface is oriented downward.

Summary of Line Surface Integrals

**Line Integrals**

**Scalar Form**

**Vector Form**

**Surface Integrals**

**Scalar Form**

**Vector Form (upward normal)**

**Surface Integrals (parametric form)**

**Scalar form**

**Vector form (upward normal)**

The Divergence Theorem

Let be a solid region bounded by a closed surface oriented by a unit normal vector directed outward from . If is a vector fielded whose component functions have continuous first partial derivatives in , then

Stokes’s Theorem

Let be an oriented surface with unit normal vector , bounded by a piecewise smooth simple closed curve with positive orientation. If is a vector field whose components functions have continuous first partial derivatives on an open region containing and , then

Summary of Integration Formulas

**Fundamental Theorem of Calculus**

**Green’s Theorem**

**Divergence Theorem**

**Fundamental Theorem of Line Integrals**

**Stokes’s Theorem**